ORDINARY DIFFERENTIAL EQUATIONS:

Solve them by → Direct Integration, Power Series Method (Know Already) → Numerical Methods (Will Study here)

EULER'S METHOD:

It is used to find the approximate solution of a first-order ODE

$$y' = f(x, y)$$

with initial condition

$$y(x_0) = y_0 \quad \text{on an interval } [x_0, x_n] \text{ or } [a, b].$$

We find the step-size $h = \cancel{\frac{x_0 - x_n}{n}} \; \frac{x_n - x_0}{n} = \frac{b-a}{n}$

and find the approximate solution values $y_1, y_2, \ldots, y_n$ at the points

$$\left.\begin{aligned} x_1 &= x_0 + h \\ x_2 &= x_0 + 2h \\ &\vdots \\ x_n &= x_0 + nh \end{aligned}\right\} \text{———} (*)$$

by computing

$$\boxed{y_{i+1} = y_i + hf(x_i, y_i)} \qquad \text{for } i = 0, 1, 2, \ldots, (n-1).$$

* Geometric Introduction:

In the interval (x_0, x_1), we approximate the curve $y(x)$ by the tangent line at (x_0, y_0) whose slope is

$$\left(\frac{dy}{dx}\right)_{(x_0, y_0)} = f(x_0, y_0)$$

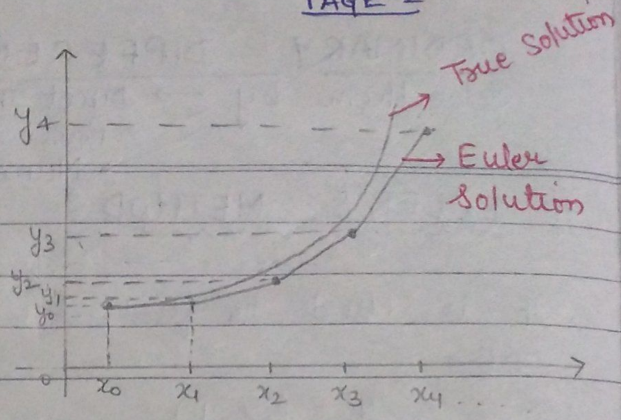
The equation of line through (x_0, y_0) whose slope is $f(x_0, y_0)$ is given by

$$y - y_0 = f(x_0, y_0)(x - x_0) \quad \text{———} (1)$$

At $(x, y) = (x_1, y_1)$, (1) gives

$$y_1 - y_0 = f(x_0, y_0)(x_1 - x_0)$$

$\because x_1 - x_0 = h$ — (Using (*))



∴ we can write

$$y_1 - y_0 = f(x_0, y_0) \cdot h$$

$$\Rightarrow y_1 = y_0 + h f(x_0, y_0) \quad \text{——— (2)}$$

Again in the interval (x_1, x_2) and through the point (x_1, y_1), we approximate the curve $y(x)$ by the tangent line at (x_1, y_1) whose slope is

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = f(x_1, y_1)$$

The equation of the tangent line is

$$y - y_1 = f(x_1, y_1) \cdot (x - x_1) \quad \text{——— (3)}$$

At $(x, y) = (x_2, y_2)$, (3) becomes

$$y_2 - y_1 = f(x_1, y_1) \cdot (x_2 - x_1)$$

$\because h = x_2 - x_1$, so,

$$y_2 - y_1 = f(x_1, y_1) \cdot h$$

$$\Rightarrow y_2 = y_1 + h f(x_1, y_1) \quad \text{——— (4)}$$

In general, going from (x_i, y_i) to (x_{i+1}, y_{i+1}) and using the fact $h = x_{i+1} - x_i$, we can write

$$y_{i+1} = y_i + h f(x_i, y_i) \quad \forall \ i = 0, 1, 2, \ldots, (n-1).$$

* Ques- Apply Euler's Method to approximate the solution of the initial value problem and calculate $y(1)$ by using $h = 0.5$:

$$\frac{dy}{dx} = x + y$$

$$y(0) = 2 \quad \text{for } 0 \leq x \leq 1.$$

Sol^n: Here, $f(x,y) = x+y$

$a = 0 = x_0$

$b = 1$

$h = 0.5$

Also, $y_0 = 2$

We know $h = \frac{b-a}{n}$

$\Rightarrow 0.5 = \frac{1-0}{n}$

$\Rightarrow n = 2$, so, $b = x_2$

For $n=1$,

$x_1 = x_0 + h$
$= 0 + 0.5$
$= 0.5$

$\therefore y_1 = y_0 + hf(x_0, y_0)$
$= 2 + (0.5)(0+2)$
$= 2 + (0.5)(2)$
$= 2+1$
$= 3$

— ($\because f(x,y) = x+y$
$\Rightarrow f(x_0, y_0) = x_0 + y_0$
$\because x_0 = 0, y_0 = 2$
$\Rightarrow f(x_0, y_0) = 0+2$
$= 2$)

For $n=2$,

$x_2 = x_1 + h$
$= 0.5 + 0.5$
$= 1$

$\therefore y_2 = y_1 + hf(x_1, y_1)$

$= 3 + (0.5)(x_1 + y_1)$
$= 3 + (0.5)(0.5+3)$
$= 3 + (0.5)(3.5)$
$= 3 + 1.75$
$= 4.75$

x_i	y_i
$x_0 = 0$	2
$x_1 = 0.5$	3
$x_2 = 1$	4.75

Table-1

$\therefore \boxed{y(1) \approx 4.75}$

$(\because y_2 \approx y(1))$

Remark:-

① The exact solution of above question is

$$y(x) = 3e^{x} - x - 1$$

② If we take $n=4$, then

$$h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$$

Now, $x_1 = x_0 + h = 0 + 0.25 = 0.25$

$y_1 = y_0 + hf(x_0, y_0) = 2 + (0.25)[0+2] = 2 + 0.5 = 2.5$

$x_2 = x_1 + h = 0.25 + 0.25 = 0.5$

$y_2 = y_1 + hf(x_1, y_1) = 2.5 + (0.25)(0.25 + 2.5) = 3.1875$

$x_3 = x_2 + h = 0.5 + 0.25 = 0.75$

$y_3 = y_2 + hf(x_2, y_2) = 3.1875 + (0.25)(0.5 + 3.1875) = 4.109375$

$x_4 = x_3 + h = 0.75 + 0.25 = 1$

$y_4 = y_3 + hf(x_3, y_3) = 4.109375 + (0.25)(0.75 + 4.109375) = 5.324219$

x_i	y_i		
	$h=0.5$	$h=0.25$	Exact value
0	2	2	2
0.5	3	3.1875	3.446164
1	4.75	5.324219	6.154845

Table-2

From this table, it is clear that the difference between the exact value and the approximate value (when $h=0.25$) is less than the difference between the exact value and approximate value (when $h=0.5$).

i.e.

$$(3.446164 - 3.1875) < (3.446164 - 3)$$
$$\Rightarrow \quad 0.258664 < 0.446164$$

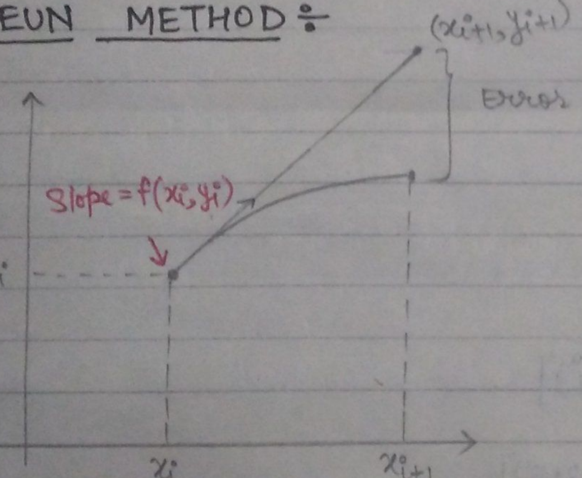
and $$(6.154845 - 5.324219) < (6.154845 - 4.75)$$
$$0.830626 < 1.404845$$

Observation ÷ The accuracy of approximate solution generated by Euler's method will improve if we decrease step-size (h).

—x—x—x—

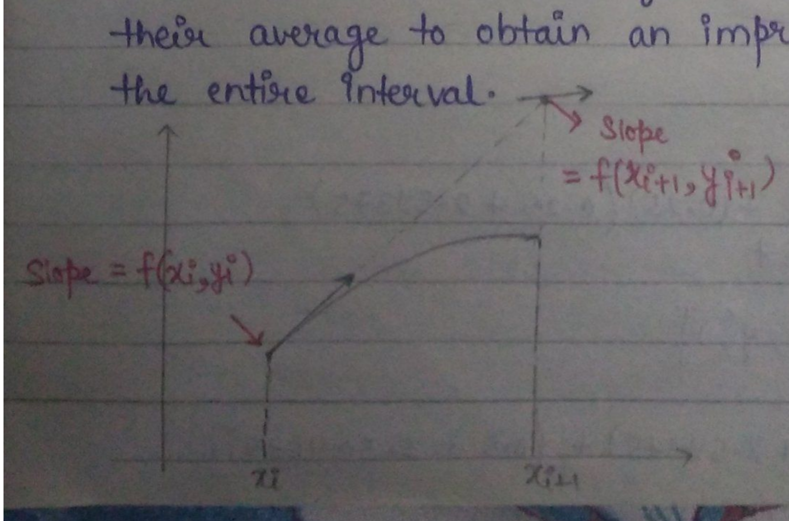
MODIFIED EULER'S METHODS ÷

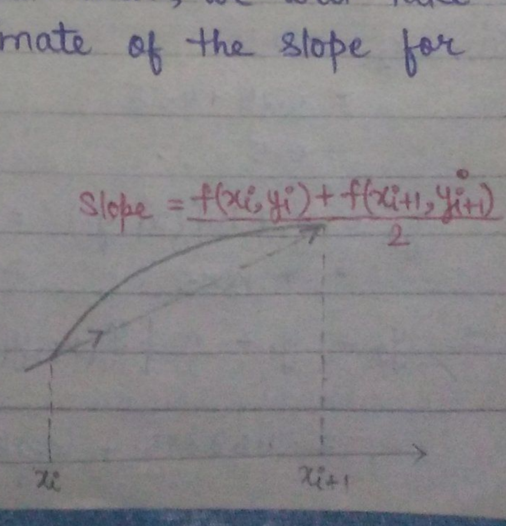
1. HEUN METHOD ÷



In Euler's method, the slope at the beginning of an interval
$$y_i' = f(x_i, y_i)$$
is used to find y_{i+1}.
i.e
$$y_{i+1} = y_i + hf(x_i, y_i)$$

Here, we will calculate slope at the beginning of the interval as well as at the end of the interval and then, we will take their average to obtain an improved estimate of the slope for the entire interval.





i.e.

take $y^{0}_{i+1} = y_i + hf(x_i, y_i)$

then ~~average~~ according to Heun's method

$$\boxed{y_{i+1} = y_i + h\left[\frac{f(x_i, y_i) + f(x_{i+1}, y^{0}_{i+1})}{2}\right]}$$

* Ques- Try the previous question with Heun's Method. (Take $h=0.25$)

Solⁿ:- We are given

$$\frac{dy}{dx} = x+y \quad ; \; y(0)=2 \quad \& \; 0 \le x \le 1$$

$\therefore f(x,y) = x+y$

$x_0 = 0, \quad y_0 = 2, \quad h = 0.25$

Now,

(i) $x_1 = x_0 + h$

$= 0 + 0.25$

$= 0.25$

$y^{0}_1 = y_0 + hf(x_0, y_0)$

$= 2.5$

$\therefore y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y^{0}_1)\right]$

$= 2 + \frac{(0.25)}{2}\left[(0+2) + (0.25 + 2.5)\right]$

$= 2.59375$

(ii) $x_2 = x_1 + h$

$= 0.25 + 0.25$

$= 0.5$

$y^{0}_2 = y_1 + hf(x_1, y_1) = 2.59375 + (0.25)(0.25 + 2.59375)$

$= \cancel{3.18475}$ $= 3.304687$

$\therefore y_2 = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y^{0}_2)\right]$

$= 2.59375 + \frac{(0.25)}{2}\left[(0.25 + 2.59375) + (0.5 + 3.304687)\right]$

$\Rightarrow y_2 = 3.424805$

(iii) $x_3 = x_2 + h$
$= 0.5 + 0.25$
$= 0.75$

$y_3^{\circ} = y_2 + hf(x_2, y_2)$
$= 3.424805 + (0.25)(0.5 + 3.424805)$
$= 4.406006$

$\therefore y_3 = y_2 + \frac{h}{2}[f(x_2, y_2) + f(x_3, y_3^{\circ})]$

$= 3.424805 + \frac{(0.25)}{2}[0.5 + 3.424805 + 0.75 + 4.406006]$

$= 4.528656$

(iv) $x_4 = x_3 + h$
$= 0.75 + 0.25$
$= 1$

$y_4^{\circ} = y_3 + hf(x_3, y_3)$
$= 4.528656 + (0.25)(0.75 + 4.528656)$
$= 5.84832$

$\therefore y_4 = y_3 + \frac{h}{2}[f(x_3, y_3) + f(x_4, y_4^{\circ})]$

$= 4.528656 + \frac{(0.25)}{2}[(0.75 + 4.528656) + (1 + 5.84832)]$

$= 6.044528$

$\therefore y(1) \approx 6.044528$

Remark:-

Clearly, from the table on the next page, it is clear that Heun's method is better than Euler's Method.
($\because$ Difference between the exact values and the values obtained by Euler's method is ~~less~~ more than the difference between the exact

x_i	y_i		
	Euler's Method	Heun's Method	Exact Value
0.25	2.5	2.59375	2.602076
0.5	3.1875	3.424805	3.446164
0.75	4.109375	4.528656	4.601000
1	5.324219	6.044528	6.154845

Table-3

values and values obtained by Heun's Method).

2. MIDPOINT METHOD :-

This technique uses Euler's Method to predict a value of y at the midpoint of the interval, i.e

$$y_{i+\frac{1}{2}} = y_i + \frac{h}{2} f(x_i, y_i)$$

Then, this predicted value is used to calculate a slope at the midpoint :-

$$y'_{i+\frac{1}{2}} = f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$$

which is assumed to represent a valid approximation of the average slope for the entire interval.
This slope is used to find y_{i+1}.
i.e

$$\boxed{y_{i+1} = y_i + hf(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})}$$

Ques :- Try the same question with midpoint method.

Soln :- Here, $f(x,y) = x+y$

$h = 0.25$

$x_0 = 0, \ y_0 = 2$

(i) $x_{\frac{1}{2}} = x_0 + \frac{h}{2}$

$= 0 + \frac{0.25}{2} = 0.125$

$$y_{\frac{1}{2}} = y_0 + \frac{h}{2} f(x_0, y_0)$$

$$= 2 + \frac{(0.25)}{2}(0+2)$$

$$= 2.25$$

$$\therefore y_1 = y_0 + h f(x_{\frac{1}{2}}, y_{\frac{1}{2}})$$

$$= 2 + (0.25)(0.125 + 2.25)$$

$$= 2.59375$$

(ii)

$$y_2 = y_1 + h f(x_{1+\frac{1}{2}}, y_{1+\frac{1}{2}})$$

where $x_{1+\frac{1}{2}} = x_1 + \frac{h}{2}$

$$= x_0 + h + \frac{h}{2}$$

$$= 0 + \frac{3}{2}(0.25)$$

$$= 0.375$$

$$\left[\begin{aligned} x_1 &= x_0 + h \\ &= 0 + 0.25 \\ &= 0.25 \end{aligned} \right.$$

$$y_{1+\frac{1}{2}} = y_1 + \frac{h}{2} f(x_1, y_1)$$

$$= 2.59375 + \frac{(0.25)}{2}(0.25 + 2.59375)$$

$$= 2.949219$$

$$\therefore y_2 = y_1 + h f(x_{1+\frac{1}{2}}, y_{1+\frac{1}{2}})$$

$$= 2.59375 + (0.25)(0.375 + 2.949219)$$

$$= 3.424805$$

(iii) $y_3 = y_2 + h f(x_{2+\frac{1}{2}}, y_{2+\frac{1}{2}})$

where $x_{2+\frac{1}{2}} = x_2 + \frac{h}{2}$

$$= 0.5 + \frac{(0.25)}{2}$$

$$= 0.5 + 0.125$$

$$= 0.625$$

$$\left(\begin{aligned} x_2 &= x_1 + h \\ &= 0.25 + 0.25 \\ &= 0.5 \end{aligned} \right.$$

$$y_{2+\frac{1}{2}} = y_2 + \frac{h}{2} f(x_2, y_2)$$
$$= 3.424805 + \frac{(0.25)}{2}(0.5 + 3.424805)$$
$$= 3.915406$$

$\therefore$ $y_3 = 3.424805 + (0.25)(0.625 + 3.915406)$
$= 4.559906$

(iv) $y_4 = y_3 + h f(x_{3+\frac{1}{2}}, y_{3+\frac{1}{2}})$

where $x_{3+\frac{1}{2}} = x_3 + \frac{h}{2}$
$= 0.75 + \frac{(0.25)}{2}$
$= 0.875$

— ($x_3 = x_2 + h$
$= 0.5 + 0.25$
$= 0.75$

$$y_{3+\frac{1}{2}} = y_3 + \frac{h}{2} f(x_3, y_3)$$
$$= 4.559906 + \frac{(0.25)}{2}(0.75 + 4.559906)$$
$$= 5.223644$$

$\therefore$ $y_4 = 4.559906 + (0.25)(0.875 + 5.223644)$
$= 6.084567$

x_i	y_i
0.25	2.59375
0.5	3.424805
0.75	4.559906
1	6.084567

Table-4

Observation:- If we compile the data of table 3 & 4, we observe that midpoint method gives more accurate values than Heun's Method.

——x——x——x——

RUNGE-KUTTA METHODS :- (RK Methods)

RK methods achieve the accuracy of Taylor series approach without requiring the calculation of higher derivatives.

The general form is

$$y_{i+1} = y_i + \phi h$$

where ϕ is called an increment function, which can be interpreted as a representative slope over the interval. The increment function can be written in general form as

$$\phi = a_1 k_1 + a_2 k_2 + \cdots + a_n k_n$$
$$= \sum_{k=1}^{n} a_j k_j$$

where the a_j's are constants are k_j's are

$$k_1 = f(x_i, y_i)$$
$$k_2 = f(x_i + p_1 h,\ y_i + q_{11} k_1 h)$$
$$k_3 = f(x_i + p_2 h,\ y_i + q_{21} k_1 h + q_{22} k_2 h)$$
$$\vdots$$
$$k_n = f(x_i + p_{n-1} h,\ y_i + q_{n-1,1} k_1 h + q_{n-1,2} k_2 h + \cdots + q_{n-1,n-1} k_{n-1} h)$$

where the p's and q's are constants.

This is called nth-order RK Method.

Special Case :- (n=1)

If $n=1$,

$$y_{i+1} = y_i + \phi h$$

where

$$\phi = a_1 k_1$$

and $\quad k_1 = f(x_i, y_i)$

$$\Rightarrow \phi = a_1 f(x_i, y_i)$$

Take $a_1 = 1$, then
$$\phi = f(x_i, y_i)$$

$\therefore$ we get
$$y_{i+1} = y_i + hf(x_i, y_i)$$
which is Euler's Method.

Note: The first-order RK method is, in fact, Euler's method.

* Second-order Runge-Kutta Methods: (When $n=2$)

The general form is

$$y_{i+1} = y_i + \phi h$$
$$= y_i + (a_1 K_1 + a_2 K_2) h \qquad (1)$$
where
$$K_1 = f(x_i, y_i)$$
$$K_2 = f(x_i + p_1 h, \; y_i + q_{11} K_1 h)$$

The values for a_1, a_2, p_1 and q_{11} are evaluated by equating eqn (1) with a second order Taylor series.

By doing this, we get
$$a_1 = 1 - a_2$$
$$p_1 = q_{11} = \frac{1}{2a_2}$$

Because we can choose an infinite number of values for a_2, there are an infinite number of second-order RK Methods.

Three of the most commonly used and preferred versions are:

(i) Huen Method without iteration :

let $a_2 = 1/2$, then

$$a_1 = 1 - a_2 = 1 - \frac{1}{2} = \frac{1}{2}$$

$$p_1 = q_{11} = \frac{1}{2(\frac{1}{2})} = 1$$

Then, eq^n ① becomes

$$y_{i+1} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_i + h, y_i + hf(x_i, y_i))\right]$$

which is Heun's Method.

[Note that K_1 is slope at the beginning of the interval
K_2 " end "]

(ii) The Mid-point Method :

Let $a_2 = 1$, then

$$a_1 = 1 - a_2 = 1 - 1 = 0$$

$$p_1 = q_{11} = \frac{1}{2(1)} = \frac{1}{2}$$

Then, eq^n ① becomes

$$y_{i+1} = y_i + K_2 h$$

where $K_1 = f(x_i, y_i)$

$$K_2 = f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2} f(x_i, y_i)\right)$$

$$\Rightarrow y_{i+1} = y_i + h\, f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2} f(x_i, y_i)\right)$$

which is Midpoint Method.

(iii) Ralston's Method :

Choose $a_2 = \frac{2}{3}$

Then, $a_1 = 1 - a_2 = 1 - \frac{2}{3} = \frac{1}{3}$

$$p_1 = q_{11} = \frac{1}{2a_2} = \frac{1}{2\left(\frac{2}{3}\right)} = \frac{3}{4}$$

Then, eqⁿ (1) becomes

$$y_{i+1} = y_i + \left(\frac{1}{3}K_1 + \frac{2}{3}K_2\right)h$$

where $K_1 = f(x_i, y_i)$

$$K_2 = f\left(x_i + \frac{3}{4}h,\ y_i + \frac{3}{4}K_1 h\right)$$

4th Order RK Method : (When $n = 4$)

As with the second-order approaches, there are an infinite number of versions of 4th-order RK Methods. The following is the most commonly used form, and therefore we call it the classical fourth-order RK Method :

$$y_{i+1} = y_i + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)h$$

where $K_1 = f(x_i, y_i)$

$$K_2 = f\left(x_i + \frac{h}{2},\ y_i + \frac{1}{2}K_1 h\right)$$

$$K_3 = f\left(x_i + \frac{h}{2},\ y_i + \frac{1}{2}K_2 h\right)$$

$$K_4 = f(x_i + h,\ y_i + K_3 h)$$

* Ques : Apply R.K fourth order to solve the initial value problem and calculate $y(0.1)$ by using $h = 0.1$

$$\frac{dy}{dx} = x^2 - y \quad ; \quad y(0) = 1.$$

Soln:- Here, $f(x, y) = x^2 - y$

$x_0 = 0$

$y_0 = 1$

$h = 0.1 \Rightarrow x_1 = 0.1$

Then, $y_1 = y(0.1)$

$= y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$

where $k_1 = f(x_0, y_0)$

$= 0^2 - 1$

$= -1$

$k_2 = f(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_1 h)$

$= f(0.05, 0.95)$

$= (0.05)^2 - 0.95$

$= 0.0025 - 0.95$

$= -0.9475$

$x_0 + \frac{h}{2} = 0 + \frac{0.1}{2}$

$= 0 + 0.05$

$= 0.05$

and

$y_0 + \frac{1}{2}k_1 h$

$= 1 + (\frac{0.1}{2})((-1))$

$= 1 - 0.05$

$= 0.95$

$k_3 = f(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_2 h)$

$= f(0.05, 0.952625)$

$= -0.950125$

$y_0 + \frac{1}{2}k_2 h$

$= 1 + (0.05)(-0.9475)$

$= 0.952625$

$k_4 = f(x_0 + h, y_0 + k_3 h)$

$= f(0.1, 0.904987)$

$= -0.894987$

$y_0 + k_3 h$

$= 1 + (0.1)(-0.950125)$

$= 0.904987$

Then,

$$y_1 = 1 + \frac{(0.1)}{6}\left[-1 + 2(-0.9475) + 2(-0.950125) + (-0.894987)\right]$$

$$\Rightarrow \boxed{y_1 = 0.905163}$$

* Ques:- Solve the initial value problem

$$\frac{dy}{dx} = -2xy^2 \quad ; \quad y(0) = 1$$

with $h = 0.2$ on the interval $[0, 0.4]$. Use the fourth-order RK Method. Compare with the exact solution.

Soln: $y(0.2) = 0.9615328$
$y(0.4) = 0.8620525$

—— x —— x —— x ——

V. Imp

FINITE DIFFERENCE ~~OF~~ METHOD FOR LINEAR ODE :-

This is used to solve an ODE which is a boundary value problem (BVP).

Consider, the general linear two point boundary value problem

$$y'' = p(x)y' + q(x)y + r(x) \quad \text{——(1)} \quad , \quad a \le x \le b$$

with boundary conditions

$$y(a) = \alpha$$
$$y(b) = \beta$$

To solve this problem using finite difference method, we divide the interval $[a, b]$ into n intervals or subintervals, so that,

$$h = \frac{(b-a)}{n}$$

To approximate the function $y(x)$ at the points

$$x_1 = a + h = x_0 + h$$
$$x_2 = a + 2h$$
$$\vdots$$
$$x_{n-1} = a + (n-1)h$$

we use the central difference formula

$$y''(x_i) \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \quad \text{——(2)}$$

$$y'(x_i) \approx \frac{y_{i+1} - y_{i-1}}{2h} \quad \text{——(3)}$$

Put ② & ③ in eqⁿ ① and writing $p(x_i)$ as p_i, $q(x_i)$ as q_i and $r(x_i)$ as r_i,

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = p_i\left(\frac{y_{i+1} - y_{i-1}}{2h}\right) + q_i y_i + r_i$$

$$\Rightarrow \frac{2(y_{i+1} - 2y_i + y_{i-1})}{2h^2} = \frac{p_i h(y_{i+1} - y_{i-1}) + q_i y_i (2h^2) + r_i(2h^2)}{2h^2}$$

$$\Rightarrow (2 + hp_i)y_{i-1} - (4 + 2h^2 q_i)y_i + (2 - hp_i)y_{i+1} = 2h^2 r_i$$

$$\forall\ i = 1, 2, \ldots, (n-1)$$

where $y(a) = \alpha = y_0$
$y(b) = \beta = y_n$

On simplifying, we will get a tridiagonal system, which can be solved.

* Ques- Use finite difference method to solve the problem

$$y'' = y + x(x-4) \quad \text{①}, \quad 0 \le x \le 4$$

with $y(0) = y(4) = 0$.

Solⁿ:- Let's take $n = 4$, so,

$$h = \frac{b-a}{n} = \frac{4-0}{4} = 1 \qquad \left[\begin{array}{l} a = 0 \\ b = 4 \end{array}\right.$$

[Sometimes, h is given in the question. If not, you can choose h arbitrarily]

Using central difference formula, we can write

$$y_i'' \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \quad \text{②}$$

Using ② in ①, we get

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = y_i + x_i(x_i - 4) \quad \text{——} (3) \qquad \text{for } i = 1, 2, 3$$

Also, $x_0 = 0$, $x_n = 4$, $y_0 = 0$, $y_4 = 0$

$x_1 = x_0 + h = 1$

$x_2 = 2$

$x_3 = 3$

For $i=1$, eq^n (3) gives

$$\frac{y_2 - 2y_1 + y_0}{(1)^2} = y_1 + x_1(x_1 - 4) \quad \text{—} (\because h = 1)$$

$\Rightarrow y_2 - 2y_1 + y_0 - y_1 = 1(1-4) = -3$

$\Rightarrow y_2 - 3y_1 = -3 \quad \text{—} (\because y_0 = 0)$

$\Rightarrow -3y_1 + y_2 = -3 \quad \text{——} (4)$

For $i=2$, eq^n (3) gives

$y_3 - 2y_2 + y_1 = y_2 + x_2(x_2 - 4)$

$\Rightarrow y_3 - 3y_2 + y_1 = 2(2-4)$

$\Rightarrow y_1 - 3y_2 + y_3 = -4 \quad \text{——} (5)$

For $i=3$, eq^n (3) gives

$y_4 - 2y_3 + y_2 = y_3 + x_3(x_3 - 4)$

$\Rightarrow y_4 - 3y_3 + y_2 = 3(3-4)$

$\Rightarrow y_2 - 3y_3 = -3 \quad \text{——} (6) \qquad \text{—} (\because y_4 = y(4) = 0)$

From ④, ⑤ and ⑥, we get

$$\left.\begin{aligned} -3y_1 + y_2 \qquad &= -3 \\ y_1 - 3y_2 + y_3 &= -4 \\ y_2 - 3y_3 &= -3 \end{aligned}\right\} \quad — (*)$$

which is a tridiagonal system.

On comparing system (*) with standard tridiagonal system, we get

$$d = [-3, -3, -3]$$
$$a = [1, 1, 0]$$
$$b = [0, 1, 1]$$
$$r = [-3, -4, -3]$$

Then, $a_1 = \dfrac{a_1}{d_1} = \dfrac{1}{-3}$

$$r_1 = \frac{r_1}{d_1} = \frac{-3}{-3} = 1$$

$$a_2 = \frac{a_2}{d_2 - b_2 a_1} = \frac{1}{(-3) - (1)\left(-\frac{1}{3}\right)} = \frac{1}{-3 + \frac{1}{3}} = \frac{3}{-8} = \frac{-3}{8}$$

$$r_2 = \frac{r_2 - b_2 r_1}{d_2 - b_2 a_1} = \frac{(-4) - (1)(1)}{(-3) - (1)\left(\frac{-1}{3}\right)} = \frac{15}{8}$$

$$r_3 = \frac{r_3 - b_3 r_2}{d_3 - b_3 a_2} = \frac{(-3) - (1)\left(\frac{15}{8}\right)}{(-3) - (1)\left(\frac{-3}{8}\right)}$$

$$= \frac{\frac{-24-15}{\cancel{8}}}{\frac{-24+3}{\cancel{8}}}$$

$$= \frac{-39}{-21}$$

$$= \frac{13}{7}$$

$\therefore$ $y_3 = r_3 = \frac{13}{7}$

$$y_2 = \cancel{y}\ r_2 - a_2 y_3$$

$$= \frac{15}{8} - \left(\frac{-3}{8}\right)\left(\frac{13}{7}\right)$$

$$= \frac{105 + 39}{8 \times 7}$$

$$= \frac{\cancel{144}\ 18}{\cancel{8} \times 7}$$

$$= \frac{18}{7}$$

$$y_1 = r_1 - a_1 y_2$$

$$= 1 - \left(\frac{-1}{3}\right)\left(\frac{18}{7}\right)$$

$$= \frac{21 + 18}{21}$$

$$= \frac{39}{21}$$

$$= \frac{13}{7}$$

You can solve this without using the technique of tridiagonal system.

$\therefore$ $y_1 = \frac{13}{7}$, $y_2 = \frac{18}{7}$, $y_3 = \frac{13}{7}$

—— x —— x ——