Mass-Energy Equivalence:

In special theory of relativity, the force acting on a particle is given by

$$F = \frac{d}{dt}(mv) = m\frac{dv}{dt} + v\frac{dm}{dt} = ma + v\frac{dm}{dt} \quad (1)$$

in which the mass 'm' of the particle depends on its velocity v and is given by

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma m_0$$, where $\gamma = (1-\frac{v^2}{c^2})^{-1/2}$.

If particle is displaced through a small distance dr along the line of action of force F, then the work done by this force on the particle is

$$dW = F.dr = \frac{d}{dt}(mv)\,dr = \frac{dr}{dt}d(mv) = v\,d(mv) \quad (2)$$

If the particle starts from rest (v = 0) and acquires velocity v under the action of the force F, then, the gain in kinetic energy will be equal to the total work done by the force F on the particle. That is

Gain in K.E. of the particle = work done by F on the particle in changing its velocity from v = 0 to v

$$Gain\ in\ K.E.(E_K) = work\ done\ by\ F\ on\ the\ particle\ in\ changing\ its\ velocity\ from\ v=0\ to\ v=v$$

$$E_K = \int_0^v dW = \int_0^v v\,d(mv) \quad (3)$$

Integrating by parts

$$E_K = v\int_0^v d(mv) - \int_0^v mv\,dv = [v.mv]_0^v - \int_0^v \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}v\,dv$$

$$= mv^2 + m_0c^2\left[\sqrt{1-\frac{v^2}{c^2}}\right]_0^v$$

$$= \frac{m_0v^2}{\sqrt{1-\frac{v^2}{c^2}}} + m_0c^2\sqrt{1-\frac{v^2}{c^2}} - m_0c^2$$

$$= \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_0c^2$$

$$\Rightarrow \quad E_K = mc^2 - m_0c^2 = (m-m_0)c^2 = \Delta m\,c^2 \quad (4)$$

This is the expression for **relativistic kinetic energy**.

This equation (4) indicates that the gain in K.E. corresponds to the relativistic increase in mass $\Delta m = m - m_0$ with increase in velocity.

In eq. (4), the quantity $m_0 c^2$ is the energy due to the rest mass of the particle and is called rest energy or proper energy 'E_0' of the particle i.e.,

$$E_0 = m_0 c^2 \tag{5}$$

Thus,

$$Total\ energy\ (E) = kinetic\ energy\ (E_K) + rest\ energy\ (E_0)$$

$$\Rightarrow \qquad E = (m - m_0)c^2 + m_0 c^2 = mc^2$$

$$\boxed{E = mc^2 = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}}}$$

This energy is called **relativistic energy** of the particle.

Equation (4) can be expressed as

$$E_K = m_0 c^2 (1 - \frac{v^2}{c^2})^{-\frac{1}{2}} - m_0 c^2$$
$$= m_0 c^2 (1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + ...) - m_0 c^2$$

If $\frac{v}{c} << 1$, then

$$E_k = m_0 c^2 (1 + \frac{1}{2}\frac{v^2}{c^2}) - m_0 c^2 = \frac{1}{2} m_0 v^2.$$

This is in agreement with the result of classical physics for kinetic energy.

Eq. (4) implies that the increase in K.E. (ΔE) of a particle is equal to the product of increase in its mass Δm and the square of the speed of light in vacuum i.e.,

$$\boxed{\Delta E = \Delta m c^2} \tag{6}$$

This equation is called **Einstein's mass-energy relation**.

According to this equation (6), an amount of energy ΔE in any form is equivalent to a mass $\Delta m = \frac{\Delta E}{c^2}$ and conversely any mass Δm is equivalent to an amount of energy $\Delta E = \Delta m c^2$. This is called the principal of equivalence of mass and energy.

Relation between Relativistic Momentum and Energy:

The relativistic momentum and energy of a particle of rest mass m_0 and moving with velocity v are respectively given as

$$p = mv = \frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma m_0 v \qquad (1)$$

$$E = mc^2 = \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma m_0 c^2 \qquad (2)$$

where $\gamma = (1-\frac{v^2}{c^2})^{-1/2}$.

$$\Rightarrow \gamma^2 = \frac{c^2}{c^2 - v^2} \qquad \Rightarrow \quad \gamma^2 v^2 = \gamma^2 c^2 - c^2 \qquad (3)$$

On squaring eq. (1), we have

$$p^2 = \gamma^2 m_0{}^2 v^2 = (\gamma^2 c^2 - c^2) m_0{}^2$$

$$\Rightarrow \quad p^2 c^2 = \gamma^2 c^4 m_0{}^2 - m_0{}^2 c^4 = E^2 - m_0{}^2 c^4$$

$$\boxed{E^2 = p^2 c^2 + m_0{}^2 c^4}$$

Transformation of Momentum and Energy:

The proper time interval $d\tau$ is related to the time interval dt observed relative to frame S, by the relation

$$dt = \frac{d\tau}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma\, d\tau \Rightarrow \frac{dt}{d\tau} = \gamma$$

Relativistic momentum: $\vec{p} = \gamma m_0 \vec{v}$

$$\Rightarrow p_x = \gamma m_0 v_x = m_0 \frac{dx}{dt}\frac{dt}{d\tau} = m_0 \frac{dx}{d\tau}$$

$$p_y = \gamma m_0 v_y = m_0 \frac{dy}{dt}\frac{dt}{d\tau} = m_0 \frac{dy}{d\tau}$$

$$p_z = \gamma m_0 v_z = m_0 \frac{dz}{dt}\frac{dt}{d\tau} = m_0 \frac{dz}{d\tau}$$

and relativistic energy: $E = mc^2 = \dfrac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma\, m_0 c^2$

$$\Rightarrow \frac{E}{c^2} = m_0 \frac{dt}{d\tau}$$

Since both rest mass and proper time interval are Lorentz invariants and hence the quantities p_x, p_y, p_z and E/c^2 get transformed from S to S' in exactly same manner as x, y, z and t transform.

$$x' = \gamma(x - vt), \quad y' = y, \quad z' = z \quad \& \quad t' = \gamma(t - \frac{v}{c^2}x)$$

Thus similar to these transformation equations, the transformation of momentum and energy are

$$p_x' = \gamma(p_x - v\frac{E}{c^2}), \quad p_y' = p_y, \quad p_z' = p_z \quad \& \quad E' = \gamma(E - v\, p_x)$$

The inverse transformations are

$$p_x = \gamma(p_x' + v\frac{E'}{c^2}), \quad p_y = p_y', \quad p_z = p_z' \quad \& \quad E = \gamma(E' + v\, p_x')$$

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References:

1. Mechanics – by Prof. D.S. Mathur; Page no. 106-160.
2. Mechanics - by J.C. Upadhyaya; Page no.91-130.

Note: For further reading and numerical problems, students are advised to read the above books given in reference.