**Theorem:** - A bounded function $f$ is integrable on $[a,b]$ iff for each $\epsilon>0$ there exist a partition $P$ of $[a,b]$ such that

$$U(f,P) - L(f,P) < \epsilon$$

**Proof:** Let $f$ be integrable on $[a,b]$ so that

$$\underline{\int_a^b} f\,dx = \overline{\int_a^b} f\,dx = \int_a^b f\,dx$$

Let $\epsilon>0$ be given.

Since the upper and lower ~~bound~~ integrals are the g.l.b and the l.u.b respectively of the upper and the lower sums, therefore, there exists partitions $P_1$ and $P_2$ of $[a,b]$ s.t

$$U(f,P_1) < \overline{\int_a^b} f\,dx + \epsilon/2 = \int_a^b f\,dx + \epsilon/2$$

and $$L(f,P_2) > \underline{\int_a^b} f\,dx - \epsilon/2 = \int_a^b f\,dx - \epsilon/2$$

Let $P = P_1 \cup P_2$

Then

$$U(f,P) \leq U(f,P_1) < \int_a^b f\,dx + \epsilon/2$$

$$< L(f,P_2) + \epsilon/2 + \epsilon/2$$

$$\leq L(f,P) + \epsilon/2 + \epsilon/2$$

$\Rightarrow U(f,P) - L(f,P) < \epsilon$ ; for a partition $P$

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Conversely,

Let $\epsilon > 0$ be given. Let $P$ be a partition of $[a,b]$ s.t

$$U(f,P) - L(f,P) < \epsilon \quad \text{—(1)}$$

Since

$$L(f,P) \le \underline{\int_a^b} f\,dx \le \overline{\int_a^b} f\,dx \le U(f,P)$$

$$\overline{\int_a^b} f\,dx - \underline{\int_a^b} f\,dx \le U(f,P) - L(f,P) < \epsilon$$

(by (1))

Since $\epsilon > 0$ be arbitrary it implies that

$$\overline{\int_a^b} f\,dx - \underline{\int_a^b} f(x) = 0$$

$$\Rightarrow \overline{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx$$

$\Rightarrow f$ is integrable.

Q. Show that the funct$^n$ $f$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is an integer} \\ 1 & \text{otherwise} \end{cases}$$

is integrable on $[0, m]$; $m$ being a positive integer.

Sol The given function is integrable because its points of discts are finite in no., being situated at $x = 1, 2 \ldots n$

Q Show that the funct$^n$ $f$ defined on $[0, 1]$ by

$$f(x) = \begin{cases} 1 & \text{if } x \neq 1/2 \\ 0 & \text{if } x = 1/2 \end{cases}$$

is integrable on $[0, 1]$ and find $\int_0^1 f\,dx$.

Sol$^n$:- The given funct$^n$ $f$ is constant on $[0, 1]$ except at $x = 1/2$

$\therefore$ $f$ is cts on $[0, 1]$ except at $x = 1/2$

So the set of points of discts of $f$ on $[0, 1]$ is finite

Hence $f$ is integrable on $[0, 1]$

To find $\int_0^1 f\,dx$; it is enough to find

$\int_0^{\bar{1}} f\,dx$ & $\int_{\underline{0}}^1 f\,dx$

calculate $\int_0^1 f\,dx$

For any partition $P = \{x_0, x_1, x_2 - - x_n\}$ of $[0,1]$

$M_i =$ l.u.b of $f$ in $[x_{i-1}, x_i] = 1 \ \forall i$

$\therefore U(f,P) = \sum_{i=1}^{n} M_i \delta_i = \sum_{i=1}^{n} \delta_i = [1-0] = 1$

This holds for every partition $P$ of $[0,1]$

$\therefore \overline{\int_0^1} f\,dx = 1$

$\text{Lly} \underline{\int_0^1} f\,dx = 1$

Hence $\int_0^1 f\,dx = 1$

Riemann Sum

Let $f$ be a real valued function defined on $[a,b]$. Let $P : x = x_0, x_1, - - x_n = b$ be a partition of $[a,b]$. Let $\xi_i$ be any arbitrary point of $\delta_i = [x_{i-1}, x_i]$ $(i=1, 2 - - n)$

Then the sum $\sum_{i=1}^{n} f(\xi_i)$ is called a Riemann Sum of $f$ on $[a,b]$ relative to $P$.

Since $\xi_i$ is arbitrary, therefore Corresponding to each Partition $P$ of $[a,b]$ there exist infinitely many Riemann Sums.

II Defⁿ: (Second Defⁿ of Integrability) (3)

A function f is said to be integrable on [a, b] if there exists a no. I s.t for each $\epsilon > 0$ $\exists$ $\delta > 0$ s.t for every partition $P: \{a = x_0, x_1, \dots x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and $\forall$ $\xi \in [x_{i-1}, x_i]$

$$\left| \sum_{i=1}^{n} f(\xi_i)\delta_i - I \right| < \epsilon$$

The number I is nothing but $\int_a^b f\, dx$

i.e A function f is Integrable over [a, b] if $\lim_{\|P\| \to 0} \sum_{i=1}^{n} f(\xi_i)\delta_i$ exists and is indepⁿ of the choice of $\delta_i$ and of the point $\xi$ of $\delta_i$. This limit I, if exists is $\int_a^b f\, dx$

Ques:- Show that $\int_1^2 f\, dx = 6$ where $f(x) = 2x + 3$.

Consider a partition

$$P = \{1 = x_0, x_1, \dots x_n = 2\}$$ of $\{1, 2\}$

dividing it into n equal sub-intervals, each of length

$$\frac{2-1}{n} = \frac{1}{n}$$ so that $\|P\| \to 0$ as $n \to \infty$

also $x_i = 1 + \frac{i}{n}$ $\quad i = 1, 2 \dots n$

$\delta_i = \frac{1}{n}$ $\quad \forall\ i = 1, 2 \dots n$

$i=1$

(4)

Consider

$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\delta i$ Take $[x_i = \xi_i]$

$= \lim_{n\to\infty}\sum_{i=1}^{n}(2x_i+3)\delta i$

$= \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left[2\left(1+\frac{i}{n}\right)+3\right]\right]\frac{1}{n}$

$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{5}{n}+\frac{2i}{n^2}\right)$

$= \lim_{n\to\infty}\left[\frac{5}{n}\cdot n + \frac{2}{n^2}\sum_{i=1}^{n} i\right]$

$= \lim_{n\to\infty}\left[5+\frac{2}{n^2}\cdot\frac{n(n+1)}{2}\right] = 5+1=6$

Since $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\delta i$ exists, the funct$^n$ is integrable

and

$$\int_1^2 f\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\delta i = 6$$

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Q Evaluate $\int_{-1}^{1} f\,dx$ ; where $f(x) = |x|$ (4)

since $f(x) = \begin{cases} -x ; & x \le 0 \\ x & x > 0 \end{cases}$

$\therefore$ f is bdd and cts on $[-1, 1]$

Hence f is integrable on $[-1, 1]$

Consider a partition

$P = \{-1 = x_0, x_1, \dots x_n = 0, x_{n+1}, x_{n+2} \dots, x_{2n} = 1\}$

divide $[-1, 1]$ into 2n equal parts (Subinterval) each of length $\frac{1}{n}$ so that

$\|P\| \to 0$ as $n \to \infty$

$$x_i = \begin{cases} -1 + \frac{i}{n} & i = 1, 2 \dots n. \\ \frac{i-n}{n} & i = n+1 \dots 2n \end{cases}$$

Take $\xi_i = x_i$

Consider

$$\lim_{n \to \infty} \sum_{i=1}^{2n} f(x_i) \delta_i$$

$$= \lim_{n \to \infty} \left[ \sum_{i=1}^{n} f(x_i) \delta_i + \sum_{i=n+1}^{2n} f(x_i) \delta_i \right]$$

$$= \lim_{n \to \infty} \left[ \sum_{i=1}^{n} -(x_i) \delta_i + \sum_{i=n+1}^{n} x_i \delta_i \right]$$

$$= \lim_{n \to \infty} \left[ \sum_{i=1}^{n} \left(1 - \frac{i}{n}\right) \frac{1}{n} + \sum_{i=n+1}^{2n} \left(\frac{i-n}{n}\right) \frac{1}{n} \right]$$

$$= \lim_{n \to \infty} \left[ n \cdot \frac{1}{n} - \frac{1}{n^2} \sum_{i=1}^{n} i + \frac{1}{n^2} \sum_{i=1}^{n} i \right]$$

$n+1 \leq i \leq 2n$

$n \leq i-1 \leq 2n$

$$\Rightarrow \int_{-1}^{1} f\,dx = \lim_{n \to \infty} \sum_{i=1}^{2n} f(x_i)\delta_i$$

$$= 1$$

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