B.A.,Sem-II,Mathematics(Algebra)
Vector Space continue..

1 Vector Space

1.1 Vector Subspace

Definition 1 *Let V be a vector space over the field F and let $W \subseteq V$. Then W will be a subspace of V if W itself is a vector space over F under the same compositions "addition of vectors" and "scalar multiplication" as in V.*

Theorem 1 *A non-empty subset W of a vector space V over a field F is a subspace of V if and only if*

1. $\alpha, \beta \in W \Rightarrow \alpha + \beta \in W$.
2. $c \in F, \alpha \in W \Rightarrow c\alpha \in W$.

In the above theorem, we need two conditions to prove any subset of a vector space to be a subspace. Both the conditions of above theorem, can be merged in a single condition through which we can also show any subset of a vector space to be a subspace.

Theorem 2 *A non-empty subset W of a vector space V over a field F is a subspace of V if and only if*

$c\alpha + d\beta \in W \quad \forall \quad c, d \in F$, *and* $\alpha, \beta \in W$.

Let us use these theorems for a subset to be a subspace of a vector space.

Example 1 *Prove that the subset $W = \{(x, 2x, 3x)|x \in R\}$ of R^3 is a subspace of R^3.*

Proof 1 *If we take $x = 0$, we see that $(0, 0, 0) \in W$, so W is non-empty. Now let $\alpha = (x, 2x, 3x)$ and $\beta = (y, 2y, 3y)$ be any two elements of W for $x \in R, y \in R$.Then*

1. $$\begin{aligned} \alpha + \beta = & \quad (x + y, 2x + 2y, 3x + 3y) \\ = & \quad ([x + y], 2[x + y], 3[x + y]) \\ = & \quad (z, 2z, 3z) \in W \quad \text{if} \quad z = x + y. \end{aligned}$$

2. $$c\alpha = (cx, 2cx, 3cx) \in W \quad \text{for any scalar} \quad c \in R.$$

   Hence by Theorem 1, W is a subspace of R^3.

Now let us try to prove W is a subspace of R^3 using Theorem 2.

Proof 2 *Let c, d be any two real numbers(scalars) and let $\alpha = (x, 2x, 3x), \beta = (y, 2y, 3y)$ be any two elements of W for some $x \in R, y \in R$.Then ,*
$c\alpha + d\beta = (cx, 2cx, 3cx) + (dy, 2dy, 3dy) = ([cx + dy], 2[cx + dy], 3[cx + dy]) \in W$
Hence W is a subspace of R^3

Note 1 *Sometimes $V_3(R)$ is used for R^3.Thus, $V_3(R)$ is a set of all triads or 3-tuples over the field R.*

Example 2 *Prove that the set of all solutions (a,b,c) of the equation $a + b + 2c = 0$ is a subspace of the vector space $V_3(R)$.*
Proof: *Let $W = \{(a, b, c) : a, b, c \in R \quad \text{and} \quad a + b + 2c = 0\}$. We see that $(0, 0, 0)$ satisfies the equation $a + b + 2c = 0$, so atleast $(0, 0, 0) \in W$ and $W \neq \phi$. Let $\alpha = (a_1, b_1, c_1)$ and $\beta = (a_2, b_2, c_2)$ be any two elements*

of W. Then

$$a_1 + b_1 + 2c_1 = 0 \quad (1)$$
$$and \quad a_2 + b_2 + 2c_2 = 0 \quad (2)$$

If a, b be any two scalars in R, we have

$$\begin{aligned} a\alpha + b\beta &= a(a_1, b_1, c_1) + b(a_2, b_2, c_2) \\ &= (aa_1, ab_1, ac_1) + (ba_2, bb_2, bc_2) \\ &= (aa_1 + ba_2, ab_1 + bb_2, ac_1 + bc_2). \end{aligned}$$

Now W will be a subspace of $V_3(R)$ if $a\alpha + b\beta \in W$ i.e. $(aa_1 + ba_2, ab_1 + bb_2, ac_1 + bc_2)$ satisfies equation $a + b + 2c = 0$.
For this,

$$\begin{aligned} (aa_1 + ba_2) + (ab_1 + bb_2) + 2(ac_1 + bc_2) &= a(a_1 + b_1 + 2c_1) + b(a_2 + b_2 + 2c_2) \\ &= a.0 + b.0 \quad [from(1)and(2)] \\ &= 0 \end{aligned}$$

$\therefore a\alpha + b\beta = (aa_1 + ba_2, ab_1 + bb_2, ac_1 + bc_2) \in W$.
Thus for $\alpha, \beta \in W$ and $a, b \in R \rightarrow a\alpha + b\beta \in W$.
Hence W is a subspace of $V_3(R)$.

Problem 1 *The set W of ordered triads $(a_1, a_2, 0)$, where $a_1, a_2 \in F$ is a subspace of $V_3(F)$.*

Problem 2 *If a_1, a_2, a_3 are fixed elements of a field F, then the set W of all ordered triads (x_1, x_2, x_3) of elements of F such that*

$$a_1x_1 + a_2x_2 + a_3x_3 = 0,$$

is a subspace of $V_3(F)$.

Theorem 3 *The intersection of any two subspaces W_1 and W_2 of a vector space $V(F)$ is also a subspace of $V(F)$.*
Proof: *Since $0 \in W_1$ and $0 \in W_2$, therefore $0 \in W_1 \cap W_2 \Rightarrow W_1 \cap W_2 \neq \phi$.*
In ordered to prove $W_1 \cap W_2$ a subspace, we need to show that for any $\alpha, \beta \in W_1 \cap W_2$ and $a, b \in F$, $a\alpha + b\beta \in W_1 \cap W_2$.
Let $\alpha, \beta \in W_1 \cap W_2 \Rightarrow \alpha, \beta \in W_1$ and $\alpha, \beta \in W_2$. Since W_1 is a subspace of a vector space $V(F)$ so for any scalars $a, b \in F$, we have $a\alpha + b\beta \in W_1$.
Similarly, W_2 is a subspace of a vector space $V(F)$, then for any scalars $a, b \in F$, we have $a\alpha + b\beta \in W_2$. This implies that $a\alpha + b\beta \in W_1 \cap W_2$. Thus for any $\alpha, \beta \in W_1 \cap W_2$ and $a, b \in F$, $a\alpha + b\beta \in W_1 \cap W_2$.
Hence $W_1 \cap W_2$ is a subspace of $V(F)$.

1.2 Linear combination of vectors

Definition 2 *Let $V(F)$ be a vector space. If $X_1, X_2, ...X_n \in V$, then any vector*

$$X = c_1X_1 + c_2X_2 + ... + c_nX_n where c_1, c_2, ..., c_n \in F$$

is called a linear combination of the vectors $X_1, X_2, ...X_n$. We also say that the vector X is generated by the vectors $X_1, X_2, ...X_n$.

1.3 Linear dependence and linear independence

Definition 3 *Let V be a vector space over the field F and let $X_1, X_2, ...X_n \in V$. We say that the vectors $X_1, X_2, ...X_n$ or the set $\{X_1, X_2, ...X_n\}$ is linearly dependent if there exist scalars $c_1, c_2, ..., c_n$ not all of them 0 (some of them may be zero)such that*

$$c_1X_1 + c_2X_2 + ... + c_nX_n = O$$

Definition 4 *If the vectors $X_1, X_2, ...X_n$ are not linearly dependent over F, then they are said to be linearly independent over F.*

Example 3 *Prove that if two vectors are linearly dependent , one of them is a scalar multiple of the other.*

or

Prove that the set $\{X_1, X_2\}$ *of two vectors is linearly dependent if* X_1 *and* X_2 *are collinear.*
Proof: *Let* X_1 *and* X_2 *are linearly dependent. Then there exist scalars* c_1 *and* c_2 *, (not both zero) such that*

$$c_1X_1 + c_2X_2 = O$$

If $c_1 \neq 0$ *, then we can have* $X_1 = (-\frac{c_2}{c_1})X_2$*,*
If $c_2 \neq 0$ *, then we can have* $X_2 = (-\frac{c_1}{c_2})X_1$
Thus one of the vector is scalar multiple of the other.
Conversely, let out of the two vectors, one is a scalar multiple of the other. Then $X_1 = cX_2$*, for some scalar* c*. This implies that* $1X_1 - cX_2 = O$*, which shows that* X_1 *and* X_2 *are linearly dependent because both the scalars* 1 *and* c *are not zero.*

Problem 3 *The set* $\{e_1, e_2\}$*, where* $e_1 = \begin{bmatrix}1\\0\end{bmatrix}$ *and* $e_2 = \begin{bmatrix}0\\1\end{bmatrix}$ *is a linearly independent set of vectors of* R^2.

Problem 4 *Examine whether the following vectors* $(1, 2, 3), (0, 1, 2)$ *and* $(0, 0, 1)$ *are linearly dependent or linearly independent.*

Problem 5 *Are the vectors* $X_1 = (1, 0, 1, 2), X_2 = (0, 1, 1, 2)$ *and* $X_3 = (1, 1, 1, 3)$ *in* R^4 *linearly dependent or linearly independent?*

Problem 6 *Are the vectors* $X_1 = (1, 2, -1), X_2 = (1, -2, 1), X_3 = (-3, 2, -1)$ *and* $X_4 = (2, 0, 0)$ *linearly dependent or linearly independent?*

Problem 7 *The set* $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ *of vectors of vector space* $V_3(R)$ *is linearly independent.*

Problem 8 *Find the value of* x *for which the set* $\{(1, -2), (x, -4)\}$ *is linearly dependent.*

Problem 9 *Determine whether each of the following subsets of* R^3 *is linearly independent.*

i. $\{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$

ii. $\{(1, 2, 3), (2, 3, 1), (-3, -4, 1)\}$

iii. $\{(-2, 7, 0), (4, 17, 2), (5, -2, 1)\}$

iv. $\{(1, 2, 3), (2, 3, 1)\}$

Problem 10 *Determine whether each of the following subsets of* P*(set of all polynomials) is linearly independent.*

I. $\{3, x + 1, x^2, x^2 + 2x + 5\}$

II. $\{x^2, x^2 + 1\}$

III. $\{1, x^2, x^3, x^2 + 1\}$

IV. $\{(1, 2, 3), (2, 3, 1)\}$

1.4 Basis

Definition 5 *A subset* S *of a vector space* $V(F)$ *is called a basis of* $V(F)$ *if*

1. *S can generate every vector of $V(F)$*
2. *S is linearly independent.*

Example 4 *The set* $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ *of vectors form a basis of the vector space* $V_3(R)$*.This is known as the standard basis of* R^3*.*

Theorem 4 *If one basis of a vector space contains* n *vectors , then all of its basis will contain* n *vectors.*

Theorem 5 *If a basis of a vector space contains* n *vectors , then any set containing more than* n *vectors is linearly dependent.*

Theorem 6 ***(Existence of a basis)*** *Every non-zero vector space has always a basis.*

Note 2 *A vector space may have more than one basis. The zero vector space* o *has no basis.*

Problem 11 *Determine whether or not the vectors* $(1, 2, 1), (2, 1, 0), (1, -1, 2)$ *form a basis of* R^3*.*

Problem 12 *Determine whether or not the vectors* $(0, 1, 0), (1, 0, 1), (1, 1, 0)$ *form a basis of* R^3*.*

1.5 Dimention of a vector space

Definition 6 *By the dimention of a vector space $V(F)$, we mean the number of vectors in a basis of $V(F)$. It is denoted by $dim V$.*

Example 5 *The set $\{(1,0),(0,1)\}$ of vectors form a basis of the vector space R^2. The dimention of R^2 is 2 i.e. $dimR^2$=2, because the number of vectors in this basis are two.*

Example 6 *The set $\{(1,0,0),(0,1,0),(0,0,1)\}$ of vectors form a basis of the vector space R^3. The dimention of R is 3 i.e. $dimR^3$=3, because the number of vectors in this basis are three.*

Note 3 *The dimension of the zero vector space is zero because the zero vector space o has no basis.*